

# L-functions

(PARI-GP version 2.13.4)

## Characters

A character on the abelian group  $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$ , e.g. from **znstar**(**q**,1)  $\leftrightarrow (\mathbf{Z}/q \mathbf{Z})^*$  or **bnrinit**  $\leftrightarrow \text{Cl}_{\mathfrak{f}}(K)$ , is coded by  $\chi = [c_1, \dots, c_k]$  such that  $\chi(g_j) = e(c_j/d_j)$ . Our  $L$ -functions consider the attached *primitive* character.

Dirichlet characters  $\chi_q(m, \cdot)$  in Conrey labelling system are alternatively concisely coded by **Mod**(**m**,**q**). Finally, a quadratic character ( $D/\cdot$ ) can also be coded by the integer  $D$ .

## L-function Constructors

An **Ldata** is a GP structure describing the functional equation for  $L(s) = \sum_{n \geq 1} a_n n^{-s}$ .

- Dirichlet coefficients given by closure  $a : N \mapsto [a_1, \dots, a_N]$ .
- Dirichlet coefficients  $a^*(n)$  for dual  $L$ -function  $L^*$ .
- Euler factor  $A = [a_1, \dots, a_d]$  for  $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$ ,
- classical weight  $k$  (values at  $s$  and  $k - s$  are related),
- conductor  $N$ ,  $\Lambda(s) = N^{s/2} \gamma_A(s)$ ,
- root number  $\varepsilon$ ;  $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$ .
- polar part: list of  $[\beta, P_{\beta}(x)]$ .

An **Linit** is a GP structure containing an **Ldata**  $L$  and an evaluation *domain* fixing a maximal order of derivation  $m$  and bit accuracy (**realbitprecision**), together with complex ranges

- for  $L$ -function:  $R = [c, w, h]$  (coding  $|\Re z - c| \leq w$ ,  $|\Im z| \leq h$ ); or  $R = [w, h]$  (for  $c = k/2$ ); or  $R = [h]$  (for  $c = k/2$ ,  $w = 0$ ).
- for  $\theta$ -function:  $T = [\rho, \alpha]$  (for  $|t| \geq \rho$ ,  $|\arg t| \leq \alpha$ ); or  $T = \rho$  (for  $\alpha = 0$ ).

### Ldata constructors

|                                                                          |                                                             |
|--------------------------------------------------------------------------|-------------------------------------------------------------|
| Riemann $\zeta$                                                          | <b>lfuncreate</b> (1)                                       |
| Dirichlet for quadratic char. ( $D/\cdot$ )                              | <b>lfuncreate</b> ( $D$ )                                   |
| Dirichlet series $L(\chi_q(m, \cdot), s)$                                | <b>lfuncreate</b> ( <b>Mod</b> ( <b>m</b> , <b>q</b> ))     |
| Dedekind $\zeta_K$ , $K = \mathbf{Q}[x]/(T)$                             | <b>lfuncreate</b> ( <b>bnf</b> ), <b>lfuncreate</b> ( $T$ ) |
| Hecke for $\chi \bmod \mathfrak{f}$                                      | <b>lfuncreate</b> ( <b>[bnr</b> , $\chi$ ])                 |
| Artin $L$ -function                                                      | <b>lfunartin</b> ( <b>nf</b> , <b>gal</b> , $M$ , $n$ )     |
| Lattice $\Theta$ -function                                               | <b>lfunqf</b> ( $Q$ )                                       |
| From eigenform $F$                                                       | <b>lfunmf</b> ( $F$ )                                       |
| Quotients of Dedekind $\eta: \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$ | <b>lfunetaquo</b> ( $M$ )                                   |
| $L(E, s)$ , $E$ elliptic curve                                           | <b>E = ellinit</b> (...)                                    |
| $L(\text{Sym}^m E, s)$ , $E$ elliptic curve                              | <b>lfunsympow</b> ( <b>E</b> , <b>m</b> )                   |
| genus 2 curve, $y^2 + xQ = P$                                            | <b>lfungenus2</b> ( <b>[P</b> , <b>Q]</b> )                 |
| dual $L$ function $\hat{L}$                                              | <b>lfundual</b> ( $L$ )                                     |
| $L_1 \cdot L_2$                                                          | <b>lfunmul</b> ( $L_1, L_2$ )                               |
| $L_1/L_2$                                                                | <b>lfundiv</b> ( $L_1, L_2$ )                               |
| $L(s - d)$                                                               | <b>lfunshift</b> ( $L, d$ )                                 |
| $L(s) \cdot L(s - d)$                                                    | <b>lfunshift</b> ( $L, d, 1$ )                              |
| twist by Dirichlet character                                             | <b>lfuntwist</b> ( $L, \chi$ )                              |

|                                     |                                                                                       |
|-------------------------------------|---------------------------------------------------------------------------------------|
| low-level constructor               | <b>lfuncreate</b> ( <b>[a</b> , <b>a*</b> , $A, k, N, \textit{eps}$ , <b>poles]</b> ) |
| check functional equation (at $t$ ) | <b>lfuncheckfeq</b> ( $L, \{t\}$ )                                                    |

### Linit constructors

|                                                  |                                                    |
|--------------------------------------------------|----------------------------------------------------|
| initialize for $L$                               | <b>lfuninit</b> ( $L, R, \{m = 0\}$ )              |
| initialize for $\theta$                          | <b>lfunthetainit</b> ( $L, \{T = 1\}, \{m = 0\}$ ) |
| cost of <b>lfuninit</b>                          | <b>lfuncost</b> ( $L, R, \{m = 0\}$ )              |
| cost of <b>lfunthetainit</b>                     | <b>lfunthetacost</b> ( $L, T, \{m = 0\}$ )         |
| Dedekind $\zeta_L$ , $L$ abelian over a subfield | <b>lfunabelianreinit</b>                           |

## L-functions

$L$  is either an **Ldata** or an **Linit** (more efficient for many values).

### Evaluate

|                                        |                                         |
|----------------------------------------|-----------------------------------------|
| $L^{(k)}(s)$                           | <b>lfun</b> ( $L, s, \{k = 0\}$ )       |
| $\Lambda^{(k)}(s)$                     | <b>lfunlambda</b> ( $L, s, \{k = 0\}$ ) |
| $\theta^{(k)}(t)$                      | <b>lfuntheta</b> ( $L, t, \{k = 0\}$ )  |
| generalized Hardy $Z$ -function at $t$ | <b>lfunhardy</b> ( $L, t$ )             |

### Zeros

|                                          |                                          |
|------------------------------------------|------------------------------------------|
| order of zero at $s = k/2$               | <b>lfunorderzero</b> ( $L, \{m = -1\}$ ) |
| zeros $s = k/2 + it$ , $0 \leq t \leq T$ | <b>lfunzeros</b> ( $L, T, \{h\}$ )       |

### Dirichlet series and functional equation

|                          |                              |
|--------------------------|------------------------------|
| $[a_n: 1 \leq n \leq N]$ | <b>lfunan</b> ( $L, N$ )     |
| conductor $N$ of $L$     | <b>lfunconductor</b> ( $L$ ) |
| root number and residues | <b>lfunrootres</b> ( $L$ )   |

### G-functions

Attached to inverse Mellin transform for  $\gamma_A(s)$ ,  $A = [a_1, \dots, a_d]$ .

|                                    |                                                  |
|------------------------------------|--------------------------------------------------|
| initialize for $G$ attached to $A$ | <b>gammamellinininit</b> ( $A$ )                 |
| $G^{(k)}(t)$                       | <b>gammamellinininv</b> ( $G, t, \{k = 0\}$ )    |
| asympt. expansion of $G^{(k)}(t)$  | <b>gammamellininvasymp</b> ( $A, n, \{k = 0\}$ ) |

Based on an earlier version by Joseph H. Silverman

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