Modular forms, modular symbols

(PARI-GP version 2.11.0)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

- a t_INT $D \equiv 0, 1 \bmod 4$: the quadratic character $(D/\cdot)$;
- a t_INTMOD Mod(m, q), $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);
- a pair $[G, chi]$, where $G = znstar(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $chi = [c_1, \ldots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	G = znstar$(q, 1)$
convert datum D to $[G, \chi]$	znchar(D)
Galois orbits of Dirichlet characters	chargalois(G)

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ; χ can be omitted: $[N, k]$ means trivial χ.

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$	mfinit$([N, k, \chi])$
initialize $S_k(\Gamma_0(N), \chi)$	mfinit$([N, k, \chi], 1)$
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$	mfinit$([N, k, \chi], 2)$
initialize $E_k(\Gamma_0(N), \chi)$	mfinit$([N, k, \chi], 3)$
initialize $M_k(\Gamma_0(N), \chi)$	mfinit$([N, k, \chi], 4)$
find eigenforms	mfsplit(M)
statistics on self-growing caches	getcache()

We let $M =$ mfinit$(\ldots)$ denote a modular space.

describe the space M	mfdescribe(M)
recover (N, k, χ)	mfparams(M)
... the space identifier (0 to 4)	mfspace(M)
... the dimension of M over $\mathbf{C}$	mfdim(M)
... a $\mathbf{C}$-basis (f_i) of M	mfbasis(M)
... a basis (F_j) of eigenforms	mfeigenbasis(M)
... polynomials defining $\mathbf{Q}(\chi)(F_j)/\mathbf{Q}(\chi)$	mffields(M)

matrix of Hecke operator T_n on (f_i)	mfheckemat(M, n)
eigenvalues of w_Q	mfatkineigenvalues(M, Q)
basis of period poynomials for weight k	mfperiodpolbasis(k)
basis of the Kohnen $+$-space	mfkohnenbasis(M)
... new space and eigenforms	mfkohneneigenbasis(M, b)
isomorphism $S_k^+(4N, \chi) \to S_{2k-1}(N, \chi^2)$	mfkohnenbijection(M)

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$	mfdim$([N, k, \chi])$
dimension of $S_k(\Gamma_0(N), \chi)$	mfdim$([N, k, \chi], 1)$
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$	mfdim$([N, k, \chi], 2)$
dimension of $M_k(\Gamma_0(N), \chi)$	mfdim$([N, k, \chi], 3)$
dimension of $E_k(\Gamma_0(N), \chi)$	mfdim$([N, k, \chi], 4)$
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	mfsturm(N, k)

$\Gamma_0(N)$ cosets

list of right $\Gamma_0(N)$ cosets	mfcosets(N)
identify coset a matrix belongs to	mftocoset

Cusps

a cusp is given by a rational number or oo.

lists of cusps of $\Gamma_0(N)$	mfcusps(N)
number of cusps of $\Gamma_0(N)$	mfnumcusps(N)
width of cusp c of $\Gamma_0(N)$	mfcuspwidth(N, c)
is cusp c regular for $M_k(\Gamma_0(N), \chi)$?	mfcuspisregular$([N, k, \chi], c)$

Create an individual modular form

Besides mfbasis and mfeigenbasis, an individual modular form can be identified by a few coefficients.

modular form from coefficients	mftobasis(mf, vec)

There are also many predefined ones:

Eisenstein series E_k on $Sl_2(\mathbf{Z})$	mfEk(k)
Eisenstein-Hurwitz series on $\Gamma_0(4)$	mfEH(k)
unary θ function (for character ψ)	mfTheta$(\{\psi\})$
Ramanujan's Δ	mfDelta()
$E_k(\chi)$	mfeisenstein(k, χ)
$E_k(\chi_1, \chi_2)$	mfeisenstein(k, χ_1, χ_2)
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	mffrometaquo(a)
newform attached to ell. curve $E/\mathbf{Q}$	mffromell(E)
identify an L-function as a eigenform	mffromlfun(L)
θ function attached to $Q > 0$	mffromqf(Q)
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$	mftraceform$([N, k, \chi])$
trace form in $S_k(\Gamma_0(N), \chi)$	mftraceform$([N, k, \chi], 1)$

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms

$f \times g$	mfmul(f, g)
f/g	mfdiv(f, g)
f^n	mfpow(f, n)
$f(q)/q^v$	mfshift(f, v)
$\sum_{i \leq k} \lambda_i F[i]$, $L = [\lambda_1, \ldots, \lambda_k]$	mflinear(F, L)
$f = g$?	mfisequal(f,g)
expanding operator $B_d(f)$	mfbd(f, d)
Hecke operator $T_n f$	mfhecke(mf, f, n)
initialize Atkin–Lehner operator w_Q	mfatkininit(mf, Q)
... apply w_Q to f	mfatkin(w_Q, f)
twist by the quadratic char $(D/\cdot)$	mftwist(f, D)
derivative wrt. $q \cdot d/dq$	mfderiv(f)
see f over an absolute field	mfreltoabs(f)
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	mfderivE2(f)
Rankin-Cohen bracket $[f, g]_n$	mfbracket(f, g, n)
Shimura lift of f for discriminant D	mfshimura(mf, f, D)

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ.

describe the form f	mfdescribe(f)
(N, k, χ) for form f	mfparams(f)
the space identifier (0 to 4) for f	mfspace(mf, f)
$[f_0, \ldots, f_n]$	mfcoefs(f, n)
f_n	mfcoef(f, n)
is f a CM form?	mfisCM(f)
Galois rep. attached to $(1, \chi)$-eigenform	mfgaloistype(M, F)
Galois rep. attached to all $(1, \chi)$ eigenforms	mfgaloistype(M)
decompose f on mfbasis(M)	mftobasis(M, f)
smallest level on which f is defined	mfconductor(M, f)
decompose f on $\oplus S_k^{\text{new}}(\Gamma_0(d))$, $d \mid N$	mftonew(M, f)
valuation of f at cusp c	mfcuspval(M, f, c)
expansion at ∞ of $f \mid_k \gamma$	mfslashexpansion(M, f, γ, n)
n-Taylor expansion of f at i	mftaylor(f, n)
all rational eigenforms matching criteria	mfeigensearch
... forms matching criteria	mfsearch

Forms embedded into C

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f.

complex embeddings of $Q(f)$	mfembed(f)
... embed coefs of f	mfembed(f, v)
evaluate f at $\tau \in \mathcal{H}$	mfeval(f, τ)
L-function attached to f	lfunmf(mf, f)
... eigenforms of new space M	lfunmf(M)

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.

initialize symbol fs attached to f	mfsymbol(M, f)
evaluate symbol fs on path p	mfsymboleval(fs, p)
Petersson product of f and g	mfpetersson(fs, gs)
period polynomial of form f	mfperiodpol(M, fs)
period polynomials for eigensymbol FS	mfmanin(FS)

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X,Y]_{k-2}$, $L_k = \mathbf{Z}[X,Y]_{k-2}$. We let $\Delta = \mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q}))$; an element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b]-[a] \to$ the path from a to b. A path is coded by the pair $[a,b]$, where a,b are rationals or `oo`, denoting the point at infinity $(1:0)$.

Let $\mathbf{M}_k(G) = \mathrm{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k-valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either $\mathbf{M}_k$ ($\varepsilon = 0$) or its $\pm$-parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$-generators (g_i) of Δ.

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$	`msinit`$(N,k,\{\varepsilon=0\})$
the level M	`msgetlevel`(M)
the weight k	`msgetweight`(M)
the sign ε	`msgetsign`(M)
Farey symbol attached to G	`mspolygon`(M)
$\mathbf{Z}[G]$-generators (g_i) and relations for Δ	`mspathgens`(M)
decompose $p=[a,b]$ on the (g_i)	`mspathlog`(M,p)

Create a symbol

Eisenstein symbol attached to cusp c	`msfromcusp`(M,c)
cuspidal symbol attached to $E/\mathbf{Q}$	`msfromell`(E)
symbol having given Hecke eigenvalues	`msfromhecke`$(M,v,\{H\})$
is s a symbol ?	`msissymbol`(M,s)

Operations on symbols

the list of all $s(g_i)$	`mseval`(M,s)
evaluate symbol s on path $p=[a,b]$	`mseval`(M,s,p)
Petersson product of s and t	`mspetersson`(M,s,t)

Operators on subspaces

An operator is given by a matrix of a fixed $\mathbf{Q}$-basis. H, if given, is a stable $\mathbf{Q}$-subspace of $\mathbf{M}_k(G)$: operator is restricted to H.

matrix of Hecke operator T_p or U_p	`mshecke`$(M,p,\{H\})$
matrix of Atkin-Lehner w_Q	`msatkinlehner`$(M,Q\{H\})$
matrix of the $*$ involution	`msstar`$(M,\{H\})$

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a $\mathbf{Q}$-basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$	`mscuspidal`(M)
Eisenstein subspace $E_k(G)^\varepsilon$	`mseisenstein`(M)
new part of $S_k(G)^\varepsilon$	`msnew`(M)
split H into simple subspaces (of dim $\leq d$)	`mssplit`$(M,H,\{d\})$
dimension of a subspace	`msdim`(M)
$(a_1,\dots,a_B)$ for attached newform	`msqexpansion`$(M,H,\{B\})$
$\mathbf{Z}$-structure from $H^1(G,L_k)$ on subspace A	`mslattice`(M,A)

Overconvergent symbols and p-adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with non-zero eigenvalue a_p, we can attach a p-adic L-function L_p. The function L_p is defined on continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle\chi\rangle^{s_1}\tau^{s_2}$, where (s_1,s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ, taking values in spaces of p-adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* $= 0$ (fastest), and that $v_p(a_p) \geq$ *flag* otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to Φ allowing to compute L_p to high accuracy.

initialize Mp to lift symbols	`mspadicinit`$(M,p,n,\{\mathit{flag}\})$
lift symbol ϕ	`mstooms`(Mp,ϕ)
eval overconvergent symbol Φ on path p	`msomseval`(Mp,Φ,p)
mu for p-adic L-functions	`mspadicmoments`$(Mp,S,\{D=1\})$
$L_p^{(r)}(\chi^s)$, $s=[s_1,s_2]$	`mspadicL`$(mu,\{s=0\},\{r=0\})$
$\hat{L}_p(\tau^i)(x)$	`mspadicseries`$(mu,\{i=0\})$