

Modular forms, modular symbols

(PARI-GP version 2.13.4)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

- a `t_INT` $D \equiv 0, 1 \bmod 4$: the quadratic character (D/\cdot) ;
- a `t_INTMOD` $\text{Mod}(m, q)$, $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);
- a pair $[G, \text{chi}]$, where $G = \text{znstar}(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \leq k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $\text{chi} = [c_1, \dots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	<code>G = znstar($q, 1$)</code>
convert datum D to $[G, \chi]$	<code>znchar(D)</code>
Galois orbits of Dirichlet characters	<code>chargalois(G)</code>

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ ; χ can be omitted: $[N, k]$ means trivial χ .

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mfinit($[N, k, \chi], 0$)</code>
initialize $S_k(\Gamma_0(N), \chi)$	<code>mfinit($[N, k, \chi], 1$)</code>
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$	<code>mfinit($[N, k, \chi], 2$)</code>
initialize $E_k(\Gamma_0(N), \chi)$	<code>mfinit($[N, k, \chi], 3$)</code>
initialize $M_k(\Gamma_0(N), \chi)$	<code>mfinit($[N, k, \chi]$)</code>
find eigenforms	<code>mfsplit(M)</code>
statistics on self-growing caches	<code>getcache()</code>

We let $M = \text{mfinit}(\dots)$ denote a modular space.	
describe the space M	<code>mfdescribe(M)</code>
recover (N, k, χ)	<code>mfparams(M)</code>
... the space identifier (0 to 4)	<code>mfspace(M)</code>
... the dimension of M over \mathbf{C}	<code>mfdim(M)</code>
... a \mathbf{C} -basis (f_i) of M	<code>mfbasis(M)</code>
... a basis (F_j) of eigenforms	<code>mfeigenbasis(M)</code>
... polynomials defining $\mathbf{Q}(\chi)/(F_j)/\mathbf{Q}(\chi)$	<code>mffields(M)</code>

matrix of Hecke operator T_n on (f_i)	<code>mfheckemat(M, n)</code>
eigenvalues of w_Q	<code>mfatkineigenvalues(M, Q)</code>
basis of period poynomials for weight k	<code>mfperiodpolbasis(k)</code>
basis of the Kohnen $+$ -space	<code>mfkohnenbasis(M)</code>
... new space and eigenforms	<code>mfkohneneigenbasis(M, b)</code>
isomorphism $S_k^+(4N, \chi) \rightarrow S_{2k-1}(N, \chi^2)$	<code>mfkohnenbijection(M)</code>

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mfdim($[N, k, \chi]$)</code>
dimension of $S_k(\Gamma_0(N), \chi)$	<code>mfdim($[N, k, \chi], 1$)</code>
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$	<code>mfdim($[N, k, \chi], 2$)</code>
dimension of $M_k(\Gamma_0(N), \chi)$	<code>mfdim($[N, k, \chi], 3$)</code>
dimension of $E_k(\Gamma_0(N), \chi)$	<code>mfdim($[N, k, \chi], 4$)</code>
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	<code>mfsturm(N, k)</code>
$\Gamma_0(N)$ cosets	
list of right $\Gamma_0(N)$ cosets	<code>mfcosets(N)</code>
identify coset a matrix belongs to	<code>mflocoset</code>

Cusps

a cusp is given by a rational number or oo.	
lists of cusps of $\Gamma_0(N)$	<code>mfcusps(N)</code>
number of cusps of $\Gamma_0(N)$	<code>mfnumcusps(N)</code>
width of cusp c of $\Gamma_0(N)$	<code>mfcuspswidth(N, c)</code>
is cusp c regular for $M_k(\Gamma_0(N), \chi)$?	<code>mfcuspsisregular($[N, k, \chi], c$)</code>

Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients	<code>mftobasis(mf, vec)</code>
There are also many predefined ones:	
Eisenstein series E_k on $Sl_2(\mathbf{Z})$	<code>mfEk(k)</code>
Eisenstein-Hurwitz series on $\Gamma_0(4)$	<code>mfEH(k)</code>
unary θ function (for character ψ)	<code>mfTheta($\{\psi\}$)</code>
Ramanujan's Δ	<code>mfDelta()</code>
$E_k(\chi)$	<code>mfeisenstein(k, χ)</code>
$E_k(\chi_1, \chi_2)$	<code>mfeisenstein(k, χ_1, χ_2)</code>
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	<code>mffrometaquo(a)</code>
newform attached to ell. curve E/\mathbf{Q}	<code>mffromell(E)</code>
identify an L -function as a eigenform	<code>mffromlfun(L)</code>
θ function attached to $Q > 0$	<code>mffromqt(Q)</code>
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$	<code>mftraceform($[N, k, \chi]$)</code>
trace form in $S_k(\Gamma_0(N), \chi)$	<code>mftraceform($[N, k, \chi], 1$)</code>

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms	
$f \times g$	<code>mfmul(f, g)</code>
f/g	<code>mfddiv(f, g)</code>
f^n	<code>mfpow(f, n)</code>
$f(q)/q^v$	<code>mfshift(f, v)</code>
$\sum_{i \leq k} \lambda_i F[i]$, $L = [\lambda_1, \dots, \lambda_k]$	<code>mflinear(F, L)</code>
$f = g?$	<code>mfisequal(f, g)</code>
expanding operator $B_d(f)$	<code>mfbd(f, d)</code>
Hecke operator $T_n f$	<code>mfhecke(mf, f, n)</code>
initialize Atkin–Lehner operator w_Q	<code>mfatkininit(mf, Q)</code>
... apply w_Q to f	<code>mfatkin(w_Q, f)</code>
twist by the quadratic char (D/\cdot)	<code>mftwist(f, D)</code>
derivative wrt. $q \cdot d/dq$	<code>mfderiv(f)</code>
see f over an absolute field	<code>mfreltoabs(f)</code>
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	<code>mfderivE2(f)</code>
Rankin-Cohen bracket $[f, g]_n$	<code>mfbracket(f, g, n)</code>
Shimura lift of f for discriminant D	<code>mfshimura(mf, f, D)</code>

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ .

describe the form f	<code>mfdescribe(f)</code>
(N, k, χ) for form f	<code>mfparams(f)</code>
the space identifier (0 to 4) for f	<code>mfspace(mf, f)</code>
$[f_0, \dots, f_n]$	<code>mfcoefs(f, n)</code>
f_n	<code>mfcoef(f, n)</code>
is f a CM form?	<code>mfisCM(f)</code>
is f an eta quotient?	<code>mfisetaquo(f)</code>
Galois rep. attached to all $(1, \chi)$ eigenforms	<code>mfgaloisistype(M)</code>
... single eigenform	<code>mfgaloisistype(M, F)</code>
... as a polynomial fixed by $\text{Ker } \rho_F$	<code>mfgaloisprojrep(M, F)</code>
decompose f on <code>mfbasis</code> (M)	<code>mftobasis(M, f)</code>
smallest level on which f is defined	<code>mfconductor(M, f)</code>
decompose f on $\oplus S_k^{\text{new}}(\Gamma_0(d))$, $d \mid N$	<code>mftonew(M, f)</code>
valuation of f at cusp c	<code>mfcuspsval(M, f, c)</code>
expansion at ∞ of $f \mid_k \gamma$	<code>mfslashexpansion(M, f, γ, n)</code>
n -Taylor expansion of f at i	<code>mftaylor(f, n)</code>
all rational eigenforms matching criteria	<code>mfeigensearch</code>
... forms matching criteria	<code>mfsearch</code>

Forms embedded into \mathbf{C}

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f .

complex embeddings of $Q(f)$	<code>mfembed(f)</code>
... embed coefs of f	<code>mfembed(f, v)</code>
evaluate f at $\tau \in \mathcal{H}$	<code>mfeval(f, τ)</code>
L -function attached to f	<code>lfunmf(mf, f)</code>
... eigenforms of new space M	<code>lfunmf(M)</code>

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.	
initialize symbol fs attached to f	<code>mfsymbol(M, f)</code>
evaluate symbol fs on path p	<code>mfsymboleval(fs, p)</code>
Petersson product of f and g	<code>mfpetersson(fs, gs)</code>
period polynomial of form f	<code>mfperiodpol(M, fs)</code>
period polynomials for eigensymbol FS	<code>mfmanin(FS)</code>

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$, $L_k = \mathbf{Z}[X, Y]_{k-2}$. We let $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$; an element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \rightarrow$ the path from a to b . A path is coded by the pair $[a, b]$, where a, b are rationals or ∞ , denoting the point at infinity ($1 : 0$).

Let $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k -valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either \mathbf{M}_k ($\varepsilon = 0$) or its \pm -parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$ -generators (g_i) of Δ .

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$
the level M
the weight k
the sign ε
Farey symbol attached to G
... attached to $H < G$
 $H \backslash G$ and right G -action
 $\mathbf{Z}[G]$ -generators (g_i) and relations for Δ
decompose $p = [a, b]$ on the (g_i)

`msinit($N, k, \{\varepsilon = 0\}$)`
`msgetlevel(M)`
`msgetweight(M)`
`msgetsign(M)`
`mspolygon(M)`
`msfarey(F, inH)`
`mscosets($genG, inH$)`
`mspathgens(M)`
`mspathlog(M, p)`

Create a symbol

Eisenstein symbol attached to cusp c
cuspidal symbol attached to E/\mathbf{Q}
symbol having given Hecke eigenvalues
is s a symbol ?

`msfromcusp(M, c)`
`msfromell(E)`
`msfromhecke($M, v, \{H\}$)`
`msissymbol(M, s)`

Operations on symbols

the list of all $s(g_i)$
evaluate symbol s on path $p = [a, b]$
Petersson product of s and t

`mseval(M, s)`
`mseval(M, s, p)`
`mspetersson(M, s, t)`

Operators on subspaces

An operator is given by a matrix of a fixed \mathbf{Q} -basis. H , if given, is a stable \mathbf{Q} -subspace of $\mathbf{M}_k(G)$: operator is restricted to H .

matrix of Hecke operator T_p or U_p
matrix of Atkin-Lehner w_Q
matrix of the $*$ involution

`mshecke($M, p, \{H\}$)`
`msatkinlehner($M, Q\{H\}$)`
`msstar($M, \{H\}$)`

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a \mathbf{Q} -basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$
Eisenstein subspace $E_k(G)^\varepsilon$
new part of $S_k(G)^\varepsilon$
split H into simple subspaces (of $\dim \leq d$)
dimension of a subspace
(a_1, \dots, a_B) for attached newform
 \mathbf{Z} -structure from $H^1(G, L_k)$ on subspace A

`mscuspidal(M)`
`mseisenstein(M)`
`msnew(M)`
`mssplit($M, H, \{d\}$)`
`msdim(M)`
`msqexpansion($M, H, \{B\}$)`
`mslattice(M, A)`

Overconvergent symbols and p -adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with nonzero eigenvalue a_p , we can attach a p -adic L -function L_p . The function L_p is defined on continuous characters of $\text{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle \chi \rangle^{s_1} \tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ , taking values in spaces of p -adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* = 0 (fastest), and that $v_p(a_p) \geq \textit{flag}$ otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions *mu* attached to Φ allowing to compute L_p to high accuracy.

initialize Mp to lift symbols
lift symbol ϕ
eval overconvergent symbol Φ on path p
 mu for p -adic L -functions
 $L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$
 $\hat{L}_p(\tau^i)(x)$

`mspadicinit($M, p, n, \{flag\}$)`
`mstooms(Mp, ϕ)`
`msomseval(Mp, Φ, p)`
`mspadicmoments($Mp, S, \{D = 1\}$)`
`mspadicL($mu, \{s = 0\}, \{r = 0\}$)`
`mspadicseries($mu, \{i = 0\}$)`

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